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COMPACT GENERALIZED HOPF AND COSYMPLECTIC SOLVMANIFOLDS AND THE HEISENBERG GROUP $H(n,1)$

BY

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ABSTRACT

In this paper we obtain a generalized Hopf structure on the total space of certain principal circle bundles over a compact cosymplectic manifold. Using this result we give new examples of compact generalized Hopf manifolds. We describe these examples as suspensions with fibre a compact quotient of the generalized Heisenberg group $H(n,1)$ by a discrete subgroup and we show an explicit realization of them as compact solvmanifolds.

1. Introduction and preliminaries

In this paper, we prove that it is possible to define a generalized Hopf structure on the total space of certain principal circle bundles over a compact cosymplectic manifold. Using this result we obtain new examples of compact generalized Hopf manifolds.

Next, we shall recall some definitions and results which be useful in the sequel.

Let M be a $2n$-dimensional **almost Hermitian manifold** with metric g and **almost complex structure** J. Denote by $\mathfrak{X}(M)$ the Lie algebra of C^∞ vector fields on M. The **Kähler 2-form** Ω is given by $\Omega(X,Y) = g(X,JY)$ and the **Lee 1-form** ω is defined by $\omega(X) = \frac{1}{(n-1)}\delta\Omega(JX)$, where δ denotes the codifferential.

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The vector field B on M given by $\omega(X) = g(X, B)$, for all $X \in \mathfrak{X}(M)$, is called the **Lee vector field** of M.

Let us recall that M is said to be **Kähler** if $[J, J] = 0$ and $d\Omega = 0$; **locally conformal Kähler** (l.c.K.) if $[J, J] = 0$, ω is closed and $d\Omega = \omega \wedge \Omega$ ([13]).

Let (M, J, g) be a l.c.K. manifold with Lee 1-form $\omega \neq 0$ at every point. (M, J, g) is said to be a **generalized Hopf (g.H.) manifold** if the Lee 1-form ω is parallel (see [14] and [15]).

The main compact non-Kähler examples of such manifolds are $S^{2n+1} \times S^1$, $n \geq 1$, and the compact nilmanifold $N(n,1) \times S^1$, where S^k is the k-dimensional unit sphere in $\mathbb{R}^{k+1}$ and $N(n,1) = \Gamma(n,1)\backslash H(n,1)$ is a compact quotient of the generalized Heisenberg group $H(n,1)$ by a discrete subgroup $\Gamma(n,1)$ (see [3], [10], [13] and [14]).

Let N be a $(2n+1)$-dimensional manifold and (φ, ξ, η, h) an **almost contact metric structure** on N. Then we have

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad h(\varphi X, \varphi Y) = h(X, Y) - \eta(X)\eta(Y),$$

for $X, Y \in \mathfrak{X}(N)$, I being the identity transformation. **The fundamental 2-form** Φ of N is defined by $\Phi(X, Y) = h(X, \varphi Y)$, for $X, Y \in \mathfrak{X}(N)$. The almost contact metric structure (φ, ξ, η, h) is said to be [1]: **Sasakian** if

$$\frac{1}{2}[\varphi, \varphi] + 2d\eta \otimes \xi = 0 \quad \text{and} \quad d\eta = \Phi;$$

cosymplectic if

$$d\eta = 0, \quad d\Phi = 0 \quad \text{and } [\varphi, \varphi] = 0. \tag{1.1}$$

We remark that on a cosymplectic manifold $(N, \varphi, \xi, \eta, h)$ the vector field ξ is parallel [1].

All the manifolds considered in this paper are assumed to be connected and of class C^∞.

2. Some principal circle bundles over a cosymplectic manifold

In this section, we shall obtain some examples of compact g.H. manifolds. These examples are principal circle bundles over certain compact cosymplectic manifolds.

We recall that there is a one-to-one correspondence between the equivalence classes of principal circle bundles over a manifold N and the cohomology group

$H^2(N,\mathbb{Z})$. Moreover, given an integral closed 2-form Φ on N, there is a principal circle bundle $\pi\colon M \to N$ with connection form θ such that Φ is the curvature form of θ (see [7]), that is,

$$d\theta = \pi^*\Phi. \tag{2.1}$$

Now, suppose that (V, J', g') is a Kähler manifold with integral Kähler 2-form Ω'. If S^1 is the unity circle then we consider on the product manifold $N = V \times S^1$ the cosymplectic structure (φ, ξ, η, h) given by

$$\begin{aligned} &\varphi = J \circ (pr_1)_*, \quad \xi = E, \quad \eta = (pr_2)^*(\theta), \\ &h = (pr_1)^*(g') + (pr_2)^*(\theta \otimes \theta) \end{aligned} \tag{2.2}$$

where $pr_1\colon N \to V$ and $pr_2\colon N \to S^1$ are the canonical projections onto the first and second factor respectively, θ is the canonical length element of S^1 and E its dual vector field. The fundamental 2-form Φ of N is $(pr_1)^*(\Omega')$.

Denote by M the total space of the principal circle bundle over N corresponding to the 2-form Φ. Then, using the results of [15], we conclude that M is a g.H. manifold. Notice that $M = S \times S^1$, S being the principal circle bundle over V corresponding to the 2-form Ω', and that the canonical examples of compact g.H. manifolds $S^{2n+1} \times S^1$ and $\Gamma(n,1)\backslash H(n,1) \times S^1$ are particular cases of this general situation. In fact, in the case of the manifold $S^{2n+1} \times S^1$ the corresponding Kähler manifold V is the n-dimensional complex projective space and in the case of the manifold $\Gamma(n,1)\backslash H(n,1) \times S^1$, V is the $2n$-dimensional real torus (see [3], [14] and [15]).

Next, we shall prove a generalization of the above result.

THEOREM 2.1: *Let $(N, \varphi, \xi, \eta, h)$ be a cosymplectic manifold with integral fundamental 2-form Φ and let $\pi\colon M \to N$ be the principal circle bundle over N corresponding to the integral closed 2-form Φ. Then M is a g.H. manifold.*

Proof: Suppose that θ is a connection form in the principal circle bundle $\pi\colon M \to N$ with curvature form Φ.

If X is a vector field on N, we shall denote by X^h the horizontal lift of X to M using the connection defined by the 1-form θ.

Let α be the length element of the circle S^1 and E its dual vector field.

We consider on M the almost Hermitian structure (J, g) given by

$$J = \varphi^h + \pi^*\eta \otimes E^* - \theta \otimes \xi^h, \quad g = \pi^*h + \theta \otimes \theta, \tag{2.3}$$

where φ^h is the horizontal lift of φ to M and E^* is the infinitesimal generator of the action of S^1 on M corresponding to E.

From (2.3), we deduce that π is a Riemannian submersion between the Riemannian manifolds (M, g) and (N, h). Thus, if X and Y are vector fields on N and $[X^h, Y^h]^h$ is the horizontal component of the vector field $[X^h, Y^h]$ with respect to the connection defined by the 1-form θ, then (see [11])

$$(2.4) \qquad [X^h, Y^h]^h = [X, Y]^h$$

and the vector field $[E^*, X^h]$ is vertical. Furthermore, using (2.1), we have that $\theta[E^*, X^h] = 0$. This implies that

$$(2.5) \qquad [E^*, X^h] = 0.$$

From (1.1), (2.1), (2.3), (2.4) and (2.5), we conclude that $[J, J] = 0$.

On the other hand, if Ω is the Kähler 2-form of M then a direct computation (see (2.3)) shows that $\Omega = \pi^*\Phi + \theta \wedge \pi^*\eta$. Therefore, by (1.1) and (2.1), we obtain that $d\Omega = \pi^*\eta \wedge \Omega$.

This proves that (M, J, g) is a l.c.K. manifold with Lee 1-form $\omega = \pi^*\eta$ and Lee vector field $B = \xi^h$.

Now, since π is a Riemannian submersion and ξ is a parallel vector field on N, we deduce that $(\nabla_{X^h} B)^h = 0$, ∇ being the Riemannian connection of the metric g and X a vector field on N (see [11]). Moreover, using (2.1), (2.3), (2.5) and the classical formula of the Riemannian connection (see [8], p. 160), we have that

$$g(\nabla_{X^h} B, E^*) = g(\nabla_{E^*} B, X^h) = -d\theta(X^h, \xi^h) = 0, \quad g(\nabla_{E^*} B, E^*) = 0.$$

Thus, $\nabla_{X^h} B = \nabla_{E^*} B = 0$, i.e., the vector field B is parallel. ∎

Remark 2.1: There exist examples of compact cosymplectic manifolds which are not topologically equivalent to the global product of a compact Kähler manifold with S^1 (see [2] and [9]).

Next, using Theorem 2.1, we shall obtain some examples of compact g.H. manifolds. For this purpose, we consider the examples of compact cosymplectic manifolds given in [9]. These examples are suspensions with fibre the $2n$-dimensional real torus $\mathbb{T}^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ of certain representations.

Let N be a compact manifold and $f\colon N \to N$ a diffeomorphism.

We consider the representation ϱ of $\mathbb{Z}$ on the group of the diffeomorphisms of $N, \mathrm{Diff}(N)$, given by

$$\varrho(k) = f^k, \tag{2.6}$$

for all $k \in \mathbb{Z}$. We define the action A of $\mathbb{Z}$ on the product manifold $N \times \mathbb{R}$ by

$$A(n,(x,z)) = (f^n(x), z-n) \tag{2.7}$$

for all $n \in \mathbb{Z}$ and $(x,z) \in N \times \mathbb{R}$. This action is free and properly discontinuous. Thus, the quotient space $M = (N \times \mathbb{R})/A$ is a compact manifold and the canonical projection $p' \colon N \times \mathbb{R} \to M$ is a covering map. Moreover, we can define a fibration τ of M on $S^1 = \mathbb{R}/\mathbb{Z}$ by $\tau[(x,z)] = [z]$, for all $(x,z) \in N \times \mathbb{R}$. It is clear that the fibers of τ are diffeomorphic to N. The space M is called the **suspension with fibre N of the representation ϱ** (see [6]).

Now, suppose that $N = \mathbb{T}^{2n}$ and that the diffeomorphism f is the Hermitian isometry $g_1 \colon (\mathbb{T}^{2n}, J, g) \to (\mathbb{T}^{2n}, J, g)$ defined by

$$g_1[(x_1,\ldots,x_n,y_1,\ldots,y_n)] = [(y_1,\ldots,y_n,-x_1,\ldots,-x_n)],$$

for all $[(x_1,\ldots,x_n,y_1,\ldots,y_n)] \in \mathbb{T}^{2n}$, where (J,g) is the natural Kähler structure on $\mathbb{T}^{2n}$. Denote by $N_1(n)$ (respectively $N_2(n)$) the suspension with fibre $\mathbb{T}^{2n}$ of the representation $\rho_1 \colon \mathbb{Z} \to \mathrm{Diff}(\mathbb{T}^{2n})$ (respectively $\rho_2 \colon \mathbb{Z} \to \mathrm{Diff}(\mathbb{T}^{2n})$) given by $\rho_1(k) = (g_1)^k$ (respectively $\rho_2(k) = (g_1)^{2k}$), for all $k \in \mathbb{Z}$. Then, the usual cosymplectic structure $(\bar\varphi, \bar\xi, \bar\eta, \bar h)$ on $\mathbb{T}^{2n} \times \mathbb{R}$ (see (2.2)) induces a cosymplectic structure $(\varphi_1, \xi_1, \eta_1, h_1)$ (respectively $(\varphi_2, \xi_2, \eta_2, h_2)$) on $N_1(n)$ (respectively $N_2(n)$). Thus, $N_1(n)$ (respectively $N_2(n)$) is a compact cosymplectic manifold (see [9]). Since the fundamental 2-form Φ_1 (respectively Φ_2) of $N_1(n)$ (respectively $N_2(n)$) is integral, we deduce that there is a principal circle bundle $\pi_1 \colon M_1(n) \to N_1(n)$ (respectively $\pi_2 \colon M_2(n) \to N_2(n)$) corresponding to the 2-form Φ_1 (respectively Φ_2). Moreover, using Theorem 2.1, we have that $M_1(n)$ (respectively $M_2(n)$) is a compact g.H. manifold.

Next, suppose that $\{X_1,\ldots,X_n,Y_1,\ldots,Y_n\}$ is the canonical global basis of vector fields on $\mathbb{T}^{2n}$ and that $\{\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n\}$ is its dual basis of 1-forms. Denote by α_i' and β_i' the 1-forms on $\mathbb{T}^{2n}$ given by

$$\alpha_i' = \alpha_i + \cos\frac{\pi}{3}\beta_i, \quad \beta_i' = -\sin\frac{\pi}{3}\beta_i.$$

If $\{X'_1, \ldots, X'_n, Y'_1, \ldots, Y'_n\}$ is the dual basis of vector fields of the basis of 1-forms $\{\alpha'_1, \ldots, \alpha'_n, \beta'_1, \ldots, \beta'_n\}$, we have on $\mathbb{T}^{2n}$ the Kähler structure (J', g') defined by

$$(2.8) \qquad J'X'_i = -Y'_i, \quad J'Y'_i = X'_i, \quad g' = \sum_{j=1}^{n}(\alpha'_j \otimes \alpha'_j + \beta'_j \otimes \beta'_j).$$

Let $g'_1\colon (\mathbb{T}^{2n}, J', g') \to (\mathbb{T}^{2n}, J', g')$ be the Hermitian isometry given by $g'_1[(x_1, \ldots, x_n, y_1, \ldots, y_n)] = [(-y_1, \ldots, -y_n, x_1 + y_1, \ldots, x_n + y_n)]$, and $N'_1(n)$ (respectively $N'_2(n)$) the suspension with fibre $\mathbb{T}^{2n}$ of the representation $\rho'_1\colon \mathbb{Z} \to \mathrm{Diff}(\mathbb{T}^{2n})$ (respectively $\rho'_2\colon \mathbb{Z} \to \mathrm{Diff}(\mathbb{T}^{2n})$) defined by $\rho'_1(k) = (g'_1)^k$ (respectively $\rho'_2(k) = (g'_1)^{2k}$), for all $k \in \mathbb{Z}$. We consider on $\mathbb{T}^{2n} \times \mathbb{R}$ the cosymplectic structure $(\bar{\varphi}', \bar{\xi}', \bar{\eta}', \bar{h}')$ given by

$$(2.9) \qquad \begin{gathered} \bar{\varphi}' = J' \circ (pr_1)_*, \qquad \bar{\xi}' = \frac{1}{c}\frac{\partial}{\partial t}, \qquad \bar{\eta}' = c\,pr_2^*(dt), \\ \bar{h}' = c^2(pr_1^*(g') + pr_2^*(dt^2)), \end{gathered}$$

where c is the real number

$$c = 2\sqrt{\frac{2}{3}\sin\frac{\pi}{3}},$$

$pr_1\colon \mathbb{T}^{2n} \times \mathbb{R} \to \mathbb{T}^{2n}$ and $pr_2\colon \mathbb{T}^{2n} \times \mathbb{R} \to \mathbb{R}$ are the canonical projections onto the first and second factor, respectively and t is the usual coordinate on $\mathbb{R}$. Then, the structure $(\bar{\varphi}', \bar{\xi}', \bar{\eta}', \bar{h}')$ induces a cosymplectic structure on $N'_1(n)$ (respectively $N'_2(n)$) (see [9]). If Φ'_1 (respectively Φ'_2) is the fundamental 2-form of $N'_1(n)$ (respectively $N'_2(n)$) and $M'_1(n)$ (respectively $M'_2(n)$) is the total space of the principal circle bundle $\pi'_1\colon M'_1(n) \to N'_1(n)$ (respectively $\pi'_2\colon M'_2(n) \to N'_2(n)$) corresponding to the integral closed 2-form Φ'_1 (respectively Φ'_2), then, using Theorem 2.1, we obtain that $M'_1(n)$ (respectively $M'_2(n)$) is a compact g.H. manifold.

Remark 2.2: If $n = 1$, the examples of compact cosymplectic manifolds $N_1(1)$, $N_2(1)$, $N'_1(1)$ and $N'_2(1)$ are not topologically equivalent to the global product of a compact Kähler manifold with S^1. Moreover, if N is a 3-dimensional compact flat orientable Riemannian manifold and its first Betti number is equal to 1, then N is diffeomorphic to one of these cosymplectic manifolds (see [2], [9] and [16]).

3. The compact g.H. manifolds $M_1(n)$, $M_2(n)$, $M'_1(n)$ and $M'_2(n)$

In this section we shall give explicit realizations of the compact g.H. manifolds $M_1(n)$, $M_2(n)$, $M'_1(n)$ and $M'_2(n)$.

We shall prove that these manifolds are suspensions with fibre a compact quotient of the **generalized Heisenberg group** $H(n,1)$ by a discrete subgroup.

We recall that $H(n,1)$ is the simply connected nilpotent Lie group of real matrices of the form

$$X = \begin{pmatrix} 1 & A & c \\ 0 & I_n & B \\ 0 & 0 & 1 \end{pmatrix} \tag{3.1}$$

where $A = (a_1, \dots, a_n)$, ${}^tB = (b_1, \dots, b_n) \in \mathbb{R}^n$ and $t \in \mathbb{R}$ (see [5]).

A global system of coordinates $(x_1, \dots, x_n, y_1, \dots, y_n, t)$ on $H(n,1)$ is defined by

$$x_i(X) = a_i, \quad y_i(X) = b_i, \quad t(X) = c, \tag{3.2}$$

with $i \in \{1, \dots, n\}$.

THE MANIFOLD $M_1(n)$. We denote by $\Gamma(n,1)$ the subgroup of matrices of $H(n,1)$ with integer entries and by $\Gamma(n,1)\backslash H(n,1)$ the space of right cosets. Then, $\Gamma(n,1)\backslash H(n,1)$ is a compact nilmanifold (see [5]). Moreover, the left invariant 1-form on $H(n,1)$

$$\widetilde{\theta} = -dt + \sum_{j=1}^{n} x_j dy_j \tag{3.3}$$

induces the 1-form $\widehat{\theta}$ on $\Gamma(n,1)\backslash H(n,1)$.

Now, we denote by $\widetilde{f}_1\colon H(n,1) \to H(n,1)$ the automorphism of $H(n,1)$ defined by

$$\widetilde{f}_1(x_1, \dots, x_n, y_1, \dots, y_n, t) = (y_1, \dots, y_n, -x_1, \dots, -x_n, t - \sum_{j=1}^{n} x_j y_j),$$

for all $(x_1, \dots, x_n, y_1, \dots, y_n, t) \in H(n,1)$. It is easy to prove that $\widetilde{f}_1$ induces a diffeomorphism $f_1\colon \Gamma(n,1)\backslash H(n,1) \to \Gamma(n,1)\backslash H(n,1)$.

Let $\overline{M}_1(n)$ be the suspension with fibre $\Gamma(n,1)\backslash H(n,1)$ of the representation $\varrho_1\colon \mathbb{Z} \to \mathrm{Diff}(\Gamma(n,1)\backslash H(n,1))$ given by $\varrho_1(k) = (f_1)^k$, for all $k \in \mathbb{Z}$. A direct computation shows that the fundamental group of $\overline{M}_1(n), \pi_1(\overline{M}_1(n))$, is the semidirect product

$$\pi_1(\overline{M}_1(n)) = \Gamma(n,1) \times_{\psi_1} \mathbb{Z}, \tag{3.4}$$

where $\psi_1\colon \mathbb{Z} \to \mathrm{Aut}(\Gamma(n,1))$ is the homomorphism of $\mathbb{Z}$ on the automorphism group of $\Gamma(n,1)$, $\mathrm{Aut}(\Gamma(n,1))$, defined by $\psi_1(k) = ((\widetilde{f}_1)_{|\Gamma(n,1)})^{-k}$ for all $k \in \mathbb{Z}$.

From (3.4), we deduce that the commutator subgroup of $\pi_1(\overline{M}_1(n))$ is

$$[\pi_1(\overline{M}_1(n)), \pi_1(\overline{M}_1(n))] = \{(p_1, \ldots, p_n, q_1, \ldots, q_n, r, 0) \in \Gamma(n,1) \times \mathbb{Z} : p_i + q_i \in 2\mathbb{Z},\ \forall i = 1, \ldots, n\}.$$

This implies that the first integral homology group $H_1(\overline{M}_1(n), \mathbb{Z})$ is $\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \overset{n)}{\cdots} \oplus \mathbb{Z}_2$.

On the other hand, the 1-form $\widehat{\theta}$ is invariant under the action A of $\mathbb{Z}$ on $\Gamma(n,1)\backslash H(n,1) \times \mathbb{R}$ defined by the diffeomorphism f_1 (see (2.7)). Consequently $\widehat{\theta}$ induces a 1-form θ_1 on $\overline{M}_1(n)$. Furthermore, if Φ_1 is the fundamental 2-form of the cosymplectic manifold $(N_1(n), \varphi_1, \xi_1, \eta_1, h_1)$ (see Section 2) then $\overline{M}_1(n)$ is a principal circle bundle on $N_1(n)$ with connection form θ_1 such that Φ_1 is the curvature form of θ_1. The projection of this bundle $\pi_1\colon \overline{M}_1(n) \to N_1(n)$ is defined by

$$\pi_1[[(x_1, \ldots, x_n, y_1, \ldots, y_n, t)], z] = [[(x_1, \ldots, x_n, y_1, \ldots, y_n)], z]$$

and the action of S^1 on $\overline{M}_1(n)$, $\phi_1\colon \overline{M}_1(n) \times S^1 \to \overline{M}_1(n)$, is given by

$$\phi_1([[(x_1, \ldots, x_n, y_1, \ldots, y_n, t)], z], [w]) = [[(x_1, \ldots, x_n, y_1, \ldots, y_n, t - w)], z]$$

for all $[[(x_1, \ldots, x_n, y_1, \ldots, y_n, t)], z] \in \overline{M}_1(n)$ and $[w] \in S^1$.

Thus, $\overline{M}_1(n)$ is diffeomorphic to $M_1(n)$.

Next, we shall show an explicit realization of the manifold $M_1(n)$ as a compact solvmanifold.

If $\widetilde{\zeta}$ denotes the vector field on $H(n,1)$ defined by

$$\widetilde{\zeta} = \frac{3\pi}{2}\left\{\sum_{j=1}^{n}\left(y_j\frac{\partial}{\partial x_j} - x_j\frac{\partial}{\partial y_j}\right) + \frac{1}{2}\sum_{j=1}^{n}(y_j^2 - x_j^2)\frac{\partial}{\partial t}\right\} \tag{3.5}$$

then its flow $\widetilde{\psi}\colon \mathbb{R} \times H(n,1) \to H(n,1)$ is given by

$$\begin{aligned}\widetilde{\psi}(z, (x_1, \ldots, x_n, y_1, \ldots, y_n, t)) = {}&(x_1\cos(\tfrac{3\pi}{2}z) + y_1\sin(\tfrac{3\pi}{2}z), \ldots,\\ &x_n\cos\left(\tfrac{3\pi}{2}z\right) + y_n\sin\left(\tfrac{3\pi}{2}z\right), y_1\cos\left(\tfrac{3\pi}{2}z\right) - x_1\sin\left(\tfrac{3\pi}{2}z\right), \ldots, y_n\cos\left(\tfrac{3\pi}{2}z\right)\\ &-x_n\sin\left(\tfrac{3\pi}{2}z\right), t + \tfrac{1}{2}\textstyle\sum_{j=1}^{n} x_jy_j(\cos 3\pi z - 1) + \tfrac{1}{4}\sum_{j=1}^{n}(y_j^2 - x_j^2)\sin 3\pi z).\end{aligned} \tag{3.6}$$

Thus, the diffeomorphism $\widetilde{\psi}(1)\colon H(n,1) \to H(n,1)$ defined by

$$\widetilde{\psi}(1)(x_1, \dots, x_n, y_1, \dots, y_n, t) = \widetilde{\psi}(1, (x_1, \dots, x_n, y_1, \dots, y_n, t))$$

is just the map $\widetilde{f}_1^{-1}$. Furthermore, for all $z \in \mathbb{R}$, the diffeomorphism $\widetilde{\psi}(z)\colon H(n,1) \to H(n,1)$ is an automorphism of $H(n,1)$. Consequently, the map $\widetilde{\psi}$ induces a Lie group homomorphism of $\mathbb{R}$ into the automorphism group of $H(n,1)$, $\mathrm{Aut}(H(n,1))$, which we also denote by $\widetilde{\psi}$.

Now, let $H(n,1) \times_{\widetilde{\psi}} \mathbb{R}$ be the semidirect product defined by the homomorphism $\widetilde{\psi}\colon \mathbb{R} \to \mathrm{Aut}(H(n,1))$. From (3.6), we deduce that a basis for the left invariant vector fields on $H(n,1) \times_{\widetilde{\psi}} \mathbb{R}$ is given by

$$X_i = \cos\left(\tfrac{3\pi}{2}z\right)\tfrac{\partial}{\partial x_i} - \sin\left(\tfrac{3\pi}{2}z\right)\tfrac{\partial}{\partial y_i} - x_i \sin\left(\tfrac{3\pi}{2}z\right)\tfrac{\partial}{\partial t}, \quad T = \tfrac{\partial}{\partial t},$$
$$Y_i = \sin\left(\tfrac{3\pi}{2}z\right)\tfrac{\partial}{\partial x_i} + \cos\left(\tfrac{3\pi}{2}z\right)\tfrac{\partial}{\partial y_i} + x_i \cos\left(\tfrac{3\pi}{2}z\right)\tfrac{\partial}{\partial t}, \quad Z = \tfrac{\partial}{\partial z},$$

for all $i \in \{1, \dots, n\}$. Then, for all $i \in \{1, \dots, n\}$,

$$[X_i, Y_i] = T, \quad [X_i, Z] = \frac{3\pi}{2} Y_i, \quad [Y_i, Z] = -\frac{3\pi}{2} X_i, \tag{3.7}$$

and the other brackets being zero. Using (3.7), we conclude that $H(n,1) \times_{\widetilde{\psi}} \mathbb{R}$ is a $(2n+2)$-dimensional simply connected solvable non-nilpotent Lie group.

On the other hand, since $\widetilde{\psi}(k)_{|\Gamma(n,1)} = \psi_1(k)$ for all $k \in \mathbb{Z}$, we obtain that the fundamental group $\pi_1(\overline{M}_1(n))$ of $\overline{M}_1(n)$ is a discrete subgroup of $H(n,1) \times_{\widetilde{\psi}} \mathbb{R}$.

Finally, it is easy to prove that the compact solvmanifold $\pi_1(\overline{M}_1(n)) \backslash (H(n,1) \times_{\widetilde{\psi}} \mathbb{R})$ is diffeomorphic to the suspension $\overline{M}_1(n)$.

Remark 3.1: 1. Let $(\widetilde{\varphi}, \widetilde{\xi}, \widetilde{\eta}, \widetilde{h})$ be the canonical Sasakian structure on $H(n,1)$ (see [4], Theorem 6.2). Then, the vector field $\widetilde{\zeta}$ given in (3.5) is an infinitesimal automorphism of the structure $(\widetilde{\varphi}, \widetilde{\xi}, \widetilde{\eta}, \widetilde{h})$, that is, $\mathfrak{L}_{\widetilde{\zeta}}\widetilde{\varphi} = 0$, $\mathfrak{L}_{\widetilde{\zeta}}\widetilde{\xi} = 0$, $\mathfrak{L}_{\widetilde{\zeta}}\widetilde{\eta} = 0$ and $\mathfrak{L}_{\widetilde{\zeta}}\widetilde{h} = 0$, $\mathfrak{L}$ being the Lie derivative operator on $H(n,1)$.

2. Since the structure $(\widetilde{\varphi}, \widetilde{\xi}, \widetilde{\eta}, \widetilde{h})$ is regular (that is, the vector field $\widetilde{\xi}$ is regular) then the space of leaves $H(n,1)/\widetilde{\xi}$ of the foliation on $H(n,1)$ defined by $\widetilde{\xi}$ is a Kähler manifold and the projection

$$\widetilde{\pi}\colon H(n,1) \to H(n,1)/\widetilde{\xi}$$

is a submersion (for more details on regular Sasakian manifolds see, for instance, [1]). In fact, in this case, $\widetilde{\xi} = \partial/\partial t$, $H(n,1) \,/\, \widetilde{\xi}$ is $\mathbb{R}^{2n}$ with the usual Kähler structure and the map $\widetilde{\pi}: : H(n,1) \to H(n,1)/\widetilde{\xi}$ is just the canonical projection $(x_1,\ldots,x_n, y_1,\ldots,y_n,t) \to (x_1,\ldots,x_n,y_1,\ldots,y_n)$. Moreover, the vector field $\widetilde{\zeta}$ is $\widetilde{\pi}$-projectable and its projection

$$\widetilde{\zeta}_* = \frac{3\pi}{2}\sum_{j=1}^{n}\left(y_j\frac{\partial}{\partial x_j} - x_j\frac{\partial}{\partial y_j}\right)$$

is an infinitesimal automorphism of the usual Kähler structure of $\mathbb{R}^{2n}$. For an extensive study of the automorphism group of a regular Sasakian manifold and its relation with the automorphism group of the corresponding Kähler manifold, we refer to [12].

3. In [9], the vector field $\widetilde{\zeta}_*$ is used in order to prove that $N_1(n)$ is a compact solvmanifold.

THE MANIFOLD $M_2(n)$. Denote by $\widetilde{f}_2$: $H(n,1) \to H(n,1)$ the automorphism of $H(n,1)$ defined by $\widetilde{f}_2 = (\widetilde{f}_1)^2$. Then $\widetilde{f}_2$ induces a diffeomorphism f_2: $\Gamma(n,1)\backslash H(n,1) \to \Gamma(n,1)\backslash H(n,1)$. In fact, $f_2 = (f_1)^2$.

Now, suppose that $\overline{M}_2(n)$ is the suspension with fibre $\Gamma(n,1)\backslash H(n,1)$ of the representation ϱ_2: $\mathbb{Z} \to \mathrm{Diff}(\Gamma(n,1)\backslash H(n,1))$ given by $\varrho_2(k) = (f_2)^k$, for all $k \in \mathbb{Z}$. The fundamental group of $\overline{M}_2(n), \pi_1(\overline{M}_2(n))$, is the semidirect product

$$\pi_1(\overline{M}_2(n)) = \Gamma(n,1) \times_{\psi_2} \mathbb{Z}, \tag{3.8}$$

where ψ_2: $\mathbb{Z} \to \mathrm{Aut}(\Gamma(n,1))$ is the homomorphism defined by $\psi_2(k) = ((\widetilde{f}_2)_{|\Gamma(n,1)})^{-k}$ for all $k \in \mathbb{Z}$.

From (3.8), we deduce that the commutator subgroup of $\pi_1(\overline{M}_2(n))$ is

$$\begin{aligned}[\pi_1(\overline{M}_2(n)), \pi_1(\overline{M}_2(n))] = \{&(p_1,\ldots,p_n,q_1,\ldots,q_n,r,0) \in \Gamma(n,1)\times\mathbb{Z} : \\ &(p_i,q_i) \in (2\mathbb{Z})^2, \forall i = 1,\ldots,n\}\end{aligned}$$

and the first integral homology group $H_1(\overline{M}_2(n),\mathbb{Z})$ is $\mathbb{Z}\oplus\mathbb{Z}_2\oplus \overset{2n)}{\cdots} \oplus\mathbb{Z}_2$.

By a similar device to that used for the manifold $M_1(n)$, we have that $\overline{M}_2(n)$ is diffeomorphic to the manifold $M_2(n)$. Furthermore, considering the vector field $2\widetilde{\zeta}$ (see (3.5)) and using the fact that $f_2 = (f_1)^2$, we obtain that $M_2(n)$ is also a compact solvmanifold.

THE MANIFOLD $M_1'(n)$. Denote by $\Gamma'(n,1)$ the discrete subgroup of $H(n,1)$ consisting of those matrices X for which $(a_1,\ldots,a_n,b_1,\ldots,b_n,c) \in (2\mathbb{Z})^{2n+1}$

(see (3.1)). Then, the left invariant 1-form on $H(n,1)$

$$\widetilde{\theta}' = \frac{1}{2}\left(dt - \sum_{j=1}^{n} x_j dy_j\right) \tag{3.9}$$

induces the 1-form $\widehat{\theta}'$ on $\Gamma'(n,1)\backslash H(n,1)$.

On the other hand, the automorphism $\widetilde{f}'_1\colon H(n,1) \to H(n,1)$ defined by

$$\begin{aligned}\widetilde{f}'_1(x_1,\ldots,x_n,y_1,\ldots,y_n,t) &= (-y_1,\ldots,-y_n,x_1+y_1,\ldots,x_n+y_n,\\ &\qquad t - \textstyle\sum_{j=1}^{n}((y_j^2/2)+x_jy_j)),\end{aligned}$$

for all $(x_1,\ldots,x_n,y_1,\ldots,y_n,t) \in H(n,1)$, induces a diffeomorphism

$$f'_1\colon \Gamma'(n,1)\backslash H(n,1) \to \Gamma'(n,1)\backslash H(n,1).$$

Let $\overline{M}'_1(n)$ be the suspension with fibre $\Gamma'(n,1)\backslash H(n,1)$ of the representation $\varrho'_1\colon \mathbb{Z} \to \mathrm{Diff}(\Gamma'(n,1)\backslash H(n,1))$ given by $\varrho'_1(k) = (f'_1)^k$, for all $k \in \mathbb{Z}$. Then we have that the fundamental group of $\overline{M}'_1(n)$, $\pi_1(\overline{M}'_1(n))$, is the semidirect product

$$\pi_1(\overline{M}'_1(n)) = \Gamma'(n,1) \times_{\psi'_1} \mathbb{Z}, \tag{3.10}$$

where $\psi'_1\colon \mathbb{Z} \to \mathrm{Aut}(\Gamma'(n,1))$ is the homomorphism defined by

$$\psi'_1(k) = ((\widetilde{f}'_1)_{|\Gamma'(n,1)})^{-k} \quad \text{for all } k \in \mathbb{Z}.$$

From (3.10), we deduce that the commutator subgroup of $\pi_1(\overline{M}'_1(n))$ is

$$[\pi_1(\overline{M}'_1(n)), \pi_1(\overline{M}'_1(n))] = \Gamma'(n,1) \times \{0\}.$$

This implies that the first integral homology group $H_1(\overline{M}'_1(n),\mathbb{Z})$ is $\mathbb{Z}$.

On the other hand, the 1-form $\widehat{\theta}'$ is invariant under the action A of $\mathbb{Z}$ on $\Gamma'(n,1)\backslash H(n,1)\times\mathbb{R}$ defined by the diffeomorphism f'_1 (see (2.7)). Consequently $\widehat{\theta}'$ induces a 1-form θ'_1 on $\overline{M}'_1(n)$. If Φ'_1 is the fundamental 2-form of the cosymplectic manifold $(N'_1(n),\varphi'_1,\xi'_1,\eta'_1,h'_1)$, then $\overline{M}'_1(n)$ is a principal circle bundle on $N'_1(n)$ with connection form θ'_1 such that Φ'_1 is the curvature form of θ'_1. The projection of this bundle $\pi'_1\colon \overline{M}'_1(n) \to N'_1(n)$ is defined by

$$\pi'_1[[(x_1,\ldots,x_n,y_1,\ldots,y_n,t)],z] = [[(\frac{x_1}{2},\ldots,\frac{x_n}{2},\frac{y_1}{2},\ldots,\frac{y_n}{2})],z]$$

and the action of S^1 on $\overline{M}'_1(n)$, $\phi'_1\colon \overline{M}'_1(n)\times S^1 \to \overline{M}'_1(n)$, is given by

$$\phi'_1([[(x_1,\ldots,x_n,y_1,\ldots,y_n,t)],z],[w]) = [[(x_1,\ldots,x_n,y_1,\ldots,y_n,t+2w)],z]$$

for all $[[(x_1,\dots,x_n,y_1,\dots,y_n,t)],z] \in \overline{M}'_1(n)$ and $[w] \in S^1$.

Thus, we conclude that $\overline{M}'_1(n)$ is diffeomorphic to the $(2n+2)$-dimensional compact g.H. manifold $M'_1(n)$.

Next, we shall describe the manifold $M'_1(n)$ as a compact solvmanifold.

Let $\widetilde{\zeta}'$ be the vector field on $H(n,1)$ defined by

$$\tag{3.11} \begin{aligned} \widetilde{\zeta}' = \frac{2\pi}{9}\sin\frac{\pi}{3}\Bigg\{ & \sum_{j=1}^{n}((2y_j+x_j)\frac{\partial}{\partial x_j} - (2x_j+y_j)\frac{\partial}{\partial y_j}) \\ & + \sum_{j=1}^{n}(y_j^2 - x_j^2)\frac{\partial}{\partial t}\Bigg\}. \end{aligned}$$

If $\widetilde{\psi}'\colon \mathbb{R}\times H(n,1)\to H(n,1)$ is the flow of $\widetilde{\zeta}'$, we have that

$$\tag{3.12} \begin{aligned} &\widetilde{\psi}'(z,(x_1,\dots,x_n,y_1,\dots,y_n,t)) = (x_1\sigma(z+1)+y_1\sigma(z),\dots, \\ &x_n\sigma(z+1)+y_n\sigma(z), -x_1\sigma(z)-y_1\sigma(z-1),\dots,-x_n\sigma(z)-y_n\sigma(z-1), \\ &t-\tfrac{2}{3}\sin\tfrac{\pi}{3}\sin(\tfrac{\pi}{3}z)\textstyle\sum_{j=1}^{n}(x_j^2\sigma(z+1)+y_j^2\sigma(z-1)+2x_jy_j\sigma(z)), \end{aligned}$$

where $\sigma\colon \mathbb{R}\to\mathbb{R}$ is the map defined by $\sigma(z) = \frac{4}{3}\sin\frac{\pi}{3}\sin(\frac{\pi}{3}z)$, for all $z\in\mathbb{R}$. Thus, the diffeomorphism $\widetilde{\psi}'(1)\colon H(n,1)\to H(n,1)$ given by

$$\widetilde{\psi}'(1)(x_1,\dots,x_n,y_1,\dots,y_n,t) = \widetilde{\psi}'(1,(x_1,\dots,x_n,y_1,\dots,y_n,t))$$

is just the map $(\widetilde{f}'_1)^{-1}$. Moreover, for all $z\in\mathbb{R}$, the diffeomorphism $\widetilde{\psi}'(z)$: $H(n,1)\to H(n,1)$ is an automorphism of $H(n,1)$. Consequently, the map $\widetilde{\psi}'$ induces a Lie group homomorphism of $\mathbb{R}$ into the group $\mathrm{Aut}(H(n,1))$ which we also denote by $\widetilde{\psi}'$.

Now, let $H(n,1)\times_{\widetilde{\psi}'}\mathbb{R}$ be the semidirect product defined by the homomorphism $\widetilde{\psi}'\colon\mathbb{R}\to\mathrm{Aut}(H(n,1))$. From (3.12), we deduce that a basis for the left invariant vector fields is given by

$$\begin{aligned} X'_i &= \sigma(z+1)\tfrac{\partial}{\partial x_i} - \sigma(z)\tfrac{\partial}{\partial y_i} + \tfrac{4}{3}x_i\sin(\tfrac{\pi}{3}z)\tfrac{\partial}{\partial t}, & T' &= -\tfrac{4}{3}\sin\tfrac{\pi}{3}\tfrac{\partial}{\partial t}, \\ Y'_i &= \sigma(z)\tfrac{\partial}{\partial x_i} - \sigma(z-1)\tfrac{\partial}{\partial y_i} + \tfrac{4}{3}x_i\sin(\tfrac{\pi}{3}(z-1))\tfrac{\partial}{\partial t}, & Z' &= \tfrac{\partial}{\partial z}, \end{aligned}$$

for all $i\in\{1,\dots,n\}$. Then, for all $i\in\{1,\dots,n\}$,

$$\tag{3.13} \begin{aligned} &[X'_i,Y'_i] = T', \quad [X'_i,Z'] = \tfrac{\pi}{3}\sin\tfrac{\pi}{3}(\tfrac{4}{3}Y'_i - \tfrac{2}{3}X'_i), \\ &[Y'_i,Z'] = \tfrac{\pi}{3}\sin\tfrac{\pi}{3}(\tfrac{2}{3}Y'_i - \tfrac{4}{3}X'_i), \end{aligned}$$

and the other brackets being zero. Using (3.13), we conclude that $H(n,1) \times_{\widetilde{\psi}'} \mathbb{R}$ is a $(2n+2)$-dimensional simply connected solvable non-nilpotent Lie group.

On the other hand, since $\widetilde{\psi}'(k)_{|\Gamma'(n,1)} = \psi_1'(k)$ for all $k \in \mathbb{Z}$, we obtain that the fundamental group $\pi_1(\overline{M}_1'(n))$ of $\overline{M}_1'(n)$ is a discrete subgroup of $H(n,1) \times_{\widetilde{\psi}'} \mathbb{R}$.

Finally, it is easy to prove that the compact solvmanifold

$$\pi_1(\overline{M}_1'(n)) \backslash (H(n,1) \times_{\widetilde{\psi}'} \mathbb{R})$$

is diffeomorphic to the suspension $\overline{M}_1'(n)$.

Remark 3.2: Let (J', g') be the Kähler structure on $\mathbb{T}^{2n}$ given by (2.8). Denote by $(\widetilde{J}', \widetilde{g}')$ the Kähler structure on $\mathbb{R}^{2n}$ induced by (J', g'). Then, we can define a regular Sasakian structure $(\widetilde{\varphi}', \widetilde{\xi}', \widetilde{\eta}', \widetilde{h}')$ on $H(n,1)$ in such a sense that the corresponding Kähler manifold $H(n,1)/\widetilde{\xi}'$ is $(\mathbb{R}^{2n}, \widetilde{J}', \widetilde{g}')$ and the projection of $H(n,1)$ onto $\mathbb{R}^{2n}$ is the canonical projection $\widetilde{\pi}$: $H(n,1) \to \mathbb{R}^{2n}$ (see Remark 3.1). The vector field $\widetilde{\zeta}'$ is an infinitesimal automorphism of the structure $(\widetilde{\varphi}', \widetilde{\xi}', \widetilde{\eta}', \widetilde{h}')$ and it is $\widetilde{\pi}$-projectable onto the vector field

$$\widetilde{\zeta}_*' = \frac{2\pi}{9} \sin \frac{\pi}{3} \left\{ \sum_{j=1}^{n} ((2y_j + x_j) \frac{\partial}{\partial x_j} - (2x_j + y_j) \frac{\partial}{\partial y_j} \right\}$$

which is an infinitesimal automorphism of the Kähler structure $(\widetilde{J}', \widetilde{g}')$ of $\mathbb{R}^{2n}$. In [9], the vector field $\widetilde{\zeta}_*'$ is used in order to prove that $N_1'(n)$ is a compact solvmanifold.

THE MANIFOLD $M_2'(n)$. Consider the automorphism $\widetilde{f}_2'$: $H(n,1) \to H(n,1)$ defined by $\widetilde{f}_2' = (\widetilde{f}_1')^2$. Then, $\widetilde{f}_2'$ induces a diffeomorphism f_2': $\Gamma'(n,1)\backslash H(n,1) \to \Gamma'(n,1)\backslash H(n,1)$. In fact, $f_2' = (f_1')^2$.

We denote by $\overline{M}_2'(n)$ the suspension with fibre $\Gamma'(n,1)\backslash H(n,1)$ of the representation ϱ_2': $\mathbb{Z} \to \mathrm{Diff}(\Gamma'(n,1)\backslash H(n,1))$ given by $\varrho_2'(k) = (f_2')^k$, for all $k \in \mathbb{Z}$. The fundamental group of $\overline{M}_2'(n)$, $\pi_1(\overline{M}_2'(n))$, is the semidirect product

$$\pi_1(\overline{M}_2'(n)) = \Gamma'(n,1) \times_{\psi_2'} \mathbb{Z}, \tag{3.14}$$

where ψ_2': $\mathbb{Z} \to \mathrm{Aut}(\Gamma'(n,1))$ is the homomorphism defined by $\psi_2'(k) = ((\widetilde{f}_2')_{|\Gamma'(n,1)})^{-k}$ for all $k \in \mathbb{Z}$.

From (3.14), we deduce that the commutator subgroup of $\pi_1(\overline{M}_2'(n))$ is

$$\begin{aligned}[\pi_1(\overline{M}_2'(n)), \pi_1(\overline{M}_2'(n))] &= \{(p_1, \ldots, p_n, q_1, \ldots, q_n, r, 0) \in \Gamma'(n,1) \times \mathbb{Z} : \\ &\qquad p_i - q_i \in 3\mathbb{Z}, \forall i = 1, \ldots, n\}.\end{aligned}$$

Thus, the first integral homology group $H_1(\overline{M}'_2(n),\mathbb{Z})$ is $\mathbb{Z}\oplus\mathbb{Z}_3\oplus\overset{n)}{\cdots}\oplus\mathbb{Z}_3$.

As in the case of the manifold $M'_1(n)$, we have that $\overline{M}'_2(n)$ is diffeomorphic to the $(2n+2)$-dimensional compact g.H. manifold $M'_2(n)$. Moreover, if we consider the vector field $2\widetilde{\zeta}'$ (see (3.11)) on $H(n,1)$, since $f'_2=(f'_1)^2$, we obtain that $M'_2(n)$ is also a compact solvmanifold.

From (3.4), (3.8), (3.10) and (3.14) we deduce that the fundamental group of the manifolds $M_i(n), M'_i(n)$ $(i=1,2)$ is not abelian. Moreover, its first Betti number is 1.

On the other hand, the first Betti number of the compact nilmanifold $\Gamma(n,1)\backslash H(n,1)\times S^1$ is $2n+1$ (see [3]).

Therefore, we conclude that

THEOREM 3.1: *The manifolds $M_1(n), M_2(n), M'_1(n)$ and $M'_2(n)$ are $(2n+2)$-dimensional compact g.H. solvmanifolds and they are not topologically equivalent to the compact g.H. manifolds $S^{2n+1}\times S^1$ and $(\Gamma(n,1)\backslash H(n,1))\times S^1$.*

4. Other examples of compact g.H. solvmanifolds

Let (M,J,g) be a Kähler manifold and (N,φ,ξ,η,h) a cosymplectic manifold. On the product manifold $M\times N$ we consider the almost contact metric structure (φ',ξ',η',h') given by

$$\varphi'=J\circ(pr_1)_*+\varphi\circ(pr_2)_*,\quad \xi'=\xi,\quad \eta'=(pr_2)^*\eta,\quad h'=(pr_1)^*g+(pr_2)^*h,$$

where $pr_1\colon M\times N\to M$ and $pr_2\colon M\times N\to N$ are the canonical projections on the first and second factor respectively. Then, $(M\times N,\varphi',\xi',\eta',h')$ is a cosymplectic manifold.

Thus, if r and n are integers, $0\le r\le n$, and on the $2r$-dimensional real torus $\mathbb{T}^{2r}$ we consider the usual Kähler structure, then the product manifolds $\mathbb{T}^{2r}\times N_i(n-r)$ and $\mathbb{T}^{2r}\times N'_i(n-r)$ with $i=1,2$ are $(2n+1)$-dimensional compact cosymplectic manifolds. Therefore, if Φ_i (respectively Φ'_i) with $i=1,2$ is the fundamental 2-form of the cosymplectic manifold $\mathbb{T}^{2r}\times N_i(n-r)$ (respectively $\mathbb{T}^{2r}\times N'_i(n-r)$), then, using Theorem 2.1, we have that the principal circle bundle $M_i(n,r)$ (respectively $M'_i(n,r)$) corresponding to the closed 2-form Φ_i (respectively Φ'_i) is a $(2n+2)$-dimensional compact g. H. manifold.

The manifolds $M_i(n,r)$ and $M'_i(n,r)$ with $i=1,2$ also are compact solvmanifolds and they can be described as suspensions with fibre a compact quotient of the generalized Heisenberg group $H(n,1)$ by a discrete subgroup.

For example, the manifold $M_1(n,r)$ is the suspension with fibre the manifold $\Gamma(n,1)\backslash H(n,1)$ of the representation $\varrho_1\colon \mathbb{Z} \to \mathrm{Diff}(\Gamma(n,1)\backslash H(n,1))$ given by $\varrho_1(k) = (f_{(1,r)})^k$, for all $k \in \mathbb{Z}$, where $f_{(1,r)}\colon \Gamma(n,1)\backslash H(n,1) \to \Gamma(n,1)\backslash H(n,1)$ is the diffeomorphism defined by

$$\begin{aligned} f_{(1,r)}[(x_1,\ldots,x_n,y_1,\ldots,y_n,t)] \;=\; & [(x_1,\ldots,x_r,y_{r+1},\ldots,y_n,y_1,\ldots,y_r,\\ & -x_{r+1},\ldots,-x_n,t-\textstyle\sum_{j=r+1}^n x_jy_j)], \end{aligned}$$

for all $[(x_1,\ldots,x_n,y_1,\ldots,y_n,t)] \in \Gamma(n,1)\backslash H(n,1)$. Using this realization of $M_1(n,r)$ we can see that the first integral homology group $H_1(M_1(n,r),\mathbb{Z})$ is $\mathbb{Z}^{2r+1} \oplus \mathbb{Z}_2 \oplus^{(n-r} \cdots \oplus \mathbb{Z}_2$.

Remark 4.1: In general, the first integral homology group of $M_i(n,r)$ (respectively, $M_i'(n,r)$) with $i=1,2$ is the first integral homology group of $\mathbb{T}^{2r}\times N_i(n-r)$ (respectively $\mathbb{T}^{2r}\times N_i'(n-r)$). Thus, the first Betti number of $M_i(n,r)$ and $M_i'(n,r)$ with $i=1,2$ is equal to $2r+1$.

Finally, in order to prove that $M_1(n,r)$ is a compact solvmanifold we consider the vector field $\widetilde{\zeta}$ on $H(n,1)$:

$$\begin{aligned} \widetilde{\zeta} \;=\; & \tfrac{3\pi}{2}\{\textstyle\sum_{j=r+1}^n (y_j\frac{\partial}{\partial x_j} - x_j\frac{\partial}{\partial y_j}) + \frac12\sum_{j=r+1}^n (y_j^2 - x_j^2)\frac{\partial}{\partial t}\}\\ & +2\pi(n-r)\{\textstyle\sum_{j=1}^r (y_j\frac{\partial}{\partial x_j} - x_j\frac{\partial}{\partial y_j}) + \frac12\sum_{j=1}^r (y_j^2 - x_j^2)\frac{\partial}{\partial t}\}. \end{aligned}$$

Then, if $\widetilde{\psi}\colon \mathbb{R}\times H(n,1) \to H(n,1)$ is the flow of $\widetilde{\zeta}$ we have that the diffeomorphism $\widetilde{\psi}(z)\colon H(n,1) \to H(n,1)$ is an automorphism of $H(n,1)$, for all $z\in\mathbb{R}$. Moreover, proceeding as in Section 3, we deduce that the semidirect product $H(n,1)\times_{\widetilde{\psi}}\mathbb{R}$ is a $(2n+2)$-dimensional simply connected solvable non-nilpotent Lie group, that the fundamental group $\pi_1(M_1(n,r))$ of $M_1(n,r)$ is a discrete subgroup of $H(n,1)\times_{\widetilde{\psi}}\mathbb{R}$ and that $M_1(n,r)$ is diffeomorphic to the compact solvmanifold $\pi_1(M_1(n,r))\backslash(H(n,1)\times_{\widetilde{\psi}}\mathbb{R})$.

References

[1] D. E. Blair, *Contact Manifolds in Riemannian Geometry*, Lecture Notes in Mathematics **509**, Springer-Verlag, Berlin, 1976.

[2] D. Chinea, M. de León and J. C. Marrero, *Topology of cosymplectic manifolds*, Journal de Mathématiques Pures et Appliquées **72** (1993), 567–591.

[3] L. A. Cordero, M. Fernández and M. de León, *Compact locally conformal Kähler nilmanifolds*, Geometriae Dedicata **21** (1986), 187–192.

[4] L. A. Cordero, M. Fernández and M. de León, *Examples of compact almost contact manifolds admitting neither Sasakian nor cosymplectic structures*, Atti del Seminario Matematico e Fisico dell'Università di Modena **34** (1985–86), 43–54.

[5] Y. Haraguchi, *Sur une généralisation des structures de contact*, Thèse, Université du Haute Alsace, Mulhouse, 1981.

[6] G. Hector and U. Hirsch, *Introduction to the Geometry of Foliations*, Part A, *Aspects of Mathematics*, Friedr. Vieweg and Sohn, 1981.

[7] S. Kobayashi, *Principal fibre bundles with 1-dimensional toroidal group*, Tôhoku Mathematical Journal **8** (1956), 29–45.

[8] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, Vol. I, Interscience, New York–London, 1963.

[9] J. C. Marrero and E. Padrón, *New examples of compact cosymplectic solvmanifolds*, preprint.

[10] J. C. Marrero and J. Rocha, *Locally conformal Kähler submersions*, Geometriae Dedicata **52** (1994), 271–289.

[11] O'Neill, *The fundamental equations of a submersion*, The Michigan Mathematical Journal **13** (1966), 459–469.

[12] S. Tanno, *The automorphism groups of almost contact Riemannian manifolds*, Tôhoku Mathematical Journal **21** (1969), 21–38.

[13] I. Vaisman, *On locally conformal almost Kähler manifolds*, Israel Journal of Mathematics **24** (1976), 338–351.

[14] I. Vaisman, *Locally conformal Kähler manifolds with parallel Lee form*, Rendiconti di Matematica, Roma **12** (1979), 263–284.

[15] I. Vaisman, *Generalized Hopf manifolds*, Geometriae Dedicata **13** (1982), 231–255.

[16] J. A. Wolf, *Spaces of Constant Curvature*, 5th ed., Publish or Perish, Inc., Wilmington, Delaware, 1984.